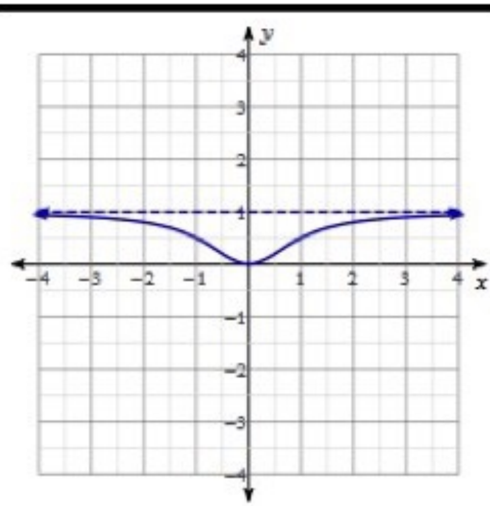
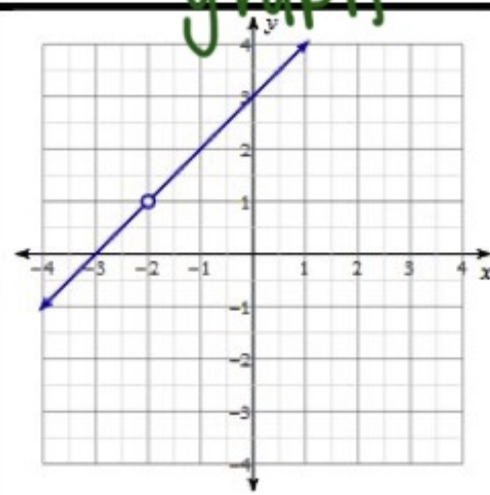
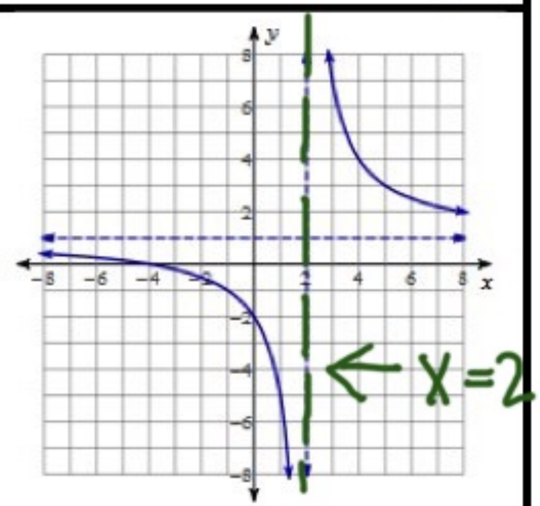
5.1 Rational Graph Properties

Find the domain for each of the rational functions below.

$y = \frac{x^2}{x^2+1}$	$y = \frac{(x+3)\cancel{(x+2)}}{\cancel{(x+2)}}$	$y = \frac{x+4}{x-2}$
Excluded Values: None	Excluded Values: $x \neq -2$	Excluded Values: $x \neq 2$
Domain: $(-\infty, \infty)$	Domain: $(-\infty,-2) \cup (-2,\infty)$	Domain: $(-\infty, 2) \cup (2,\infty)$
Continuous Graph No breaks	Removable Discontinuity one or more factors cancel → hole in the graph	Non-Removable Discontinuity Factor stays in the denominator → Vertical Asymptote
		

A discontinuity is basically A break in the domain/graph

How do you find discontinuities?

find the excluded values (set denominator equal to 0)

Holes

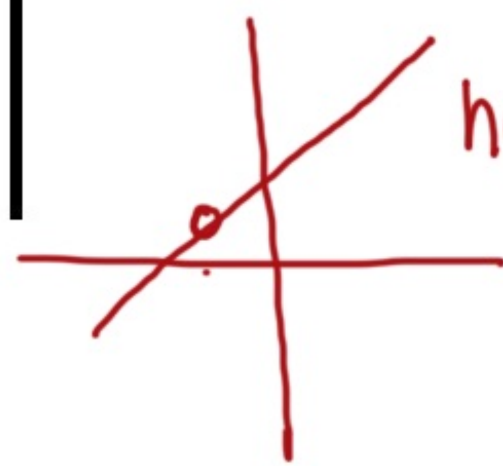
To find a hole (Removable discontinuity), we need to make sure that the function is simplified, meaning we need to factor & cancel.

- The x-coordinate is the zero from any factor that cancelled out. (excluded value)
- To find the y-coordinate, we plug the x-coordinate into the simplified function.

Ex: $f(x) = \frac{(x+5)\cancel{(x+2)}}{\cancel{x+2}}$

$f(x) = x+5, \; x \neq -2$

$f(-2) = -2+5 = 3$



hole: $(-2, 3)$

Vertical Asymptotes

Vertical asymptotes (non-removable) appear in the denominator of a **simplified** rational function. These vertical lines are written in the form $x = k$, where *k* is a constant.

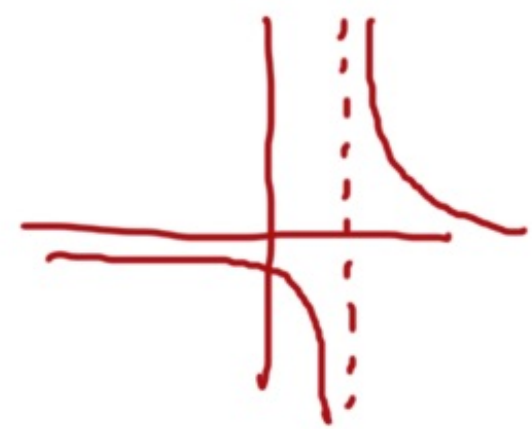
- To find the location of the asymptote, find the *x*-values that make the denominator equal to 0.

* come from factors that DON'T cancel on bottom

Ex: $f(x) = \frac{1}{x-2}$

$x - 2 = 0$

$\boxed{\text{VA: } x = 2}$



Horizontal (Behavioral) Asymptotes (end behavior)

To find a horizontal asymptote of a graph of a rational function, compare the degree of the numerator (m) with the degree of the denominator (n).

- If $m < n$ (bottom heavy), the graph has horizontal asymptote at $y = 0$.
- If $m > n$ (top heavy), the graph has no horizontal asymptote.
- If $m = n$ (equally weighted), you get the horizontal asymptote by dividing the leading coefficient of the numerator (a) by the leading coefficient of the denominator (b).

$y = \frac{a}{b}$

Ex: $f(x) = \frac{1x^2}{3x^2-12}$

degrees equal

HA: $\boxed{y = \frac{1}{3}}$

Ex: $f(x) = \frac{x+1}{2x^2+2}$

bottom heavy

$\boxed{\text{HA: } y = 0}$

EXAMPLE

State the domain for each function. State any discontinuities and identify any asymptotes.

$-1 \times -3 = 3$
$-1 + -3 = -4$

A. $y = \dfrac{x+3}{x^2-4x+3} = \dfrac{x+3}{(x-1)(x-3)}, \; x \neq 1, 3$

<u>Domain:</u>

$(-\infty, 1) \cup (1, 3) \cup (3, \infty]$

<u>Holes:</u>

None

<u>VA:</u>

$x = 1, \; x = 3$ (excluded values that didn't cancel out)

<u>HA:</u>

$y = 0$

B. $y = \frac{x-5}{x^2+1}$ (No excluded values)

Domain:

$(-\infty, \infty)$

Holes:

None

VA:

None

HA:

$y=0$

C. $y = \frac{x^2 - 3x - 4}{x - 4}$ $\begin{array}{ccc} -4 & \times & 1 \;\; -4 \\ -4 & + & 1 \;\; -3 \end{array}$ $\frac{\cancel{(x-4)}(x+1)}{\cancel{x-4}} = x+1, x \neq 4$

Domain:

$(-\infty, 4) \cup (4, \infty)$

Holes:

(Plug $x=4$ into simplified) $f(4) = 4+1 = 5$

$\boxed{(4,5)}$

VA:

None

HA:

None

D. $y = \frac{1}{x^2 - 16} = \frac{1}{(x-4)(x+4)}$, $x \neq 4, -4$

Domain:

$(-\infty, -4) \cup (-4, 4) \cup (4, \infty)$

Holes:

None

VA:

$x = 4$, $x = -4$

HA:

$y = 0$

E. $y = \frac{x+1}{x^2-5x-6} = \frac{\cancel{(x+1)}}{(x-6)\cancel{(x+1)}} = \frac{1}{x-6}, x \neq 6, -1$

Domain:

$(-\infty, -1) \cup (-1, 6) \cup (6, \infty)$

Holes:

$x = -1$ $f(-1) = \frac{1}{-1-6} = \frac{1}{-7}$ $\boxed{\left(-1, -\frac{1}{7}\right)}$

VA:

$x = 6$

☆ A number can't be both a hole & a VA.

HA:

$y = 0$

F. $y = \frac{x^2+2x}{x+2}$ $\frac{x\cancel{(x+2)}}{\cancel{x+2}} = x, \; x \neq -2$

Domain:

$(-\infty, -2) \cup (-2, \infty)$

Holes:

$x = -2$

$f(-2) = -2$ $(-2, -2)$

VA:

None

HA:

None

To find *x*-intercepts:

set numerator of SIMPLIFIED function = 0

To find *y*-intercepts:

plug 0 in for x. $\left(\dfrac{\text{constant of numerator}}{\text{constant of denominator}}\right)$

EXAMPLE **Find all *x* and *y*-intercepts for each function.**

G. $y = \frac{x-3}{x+5}$

x-intercept(s):

$x - 3 = 0 \rightarrow (3, 0)$

y-intercept:

$f(0) = \frac{0-3}{0+5}$ $\quad (0, -3/5)$

H. $y = \frac{x-3}{x^2+5x+6}$ $\quad 3 \times 2 \; 6$, $3 + 2 \; 5$ $\quad \frac{x-3}{(x+3)(x+2)}$

x-intercept(s):

$(3, 0)$

y-intercept: $\left(0, -\frac{3}{6}\right) \rightarrow \boxed{\left(0, -\frac{1}{2}\right)}$

<table>
<tr><th>Find…</th><th colspan="2">By…</th></tr>
<tr><td rowspan="3">Discontinuities</td><td colspan="2">values that make the denominator equal to 0</td></tr>
<tr><td>Removable</td><td>Non-Removable</td></tr>
<tr><td>Factors that cancel → hole</td><td>Factors that stay → Vertical Asymptote</td></tr>
<tr><td>Holes</td><td colspan="2">Plug x from cancelled factor into simplified function to find y.</td></tr>
<tr><td>Vertical Asymptotes</td><td colspan="2">x of the non cancelled factor.</td></tr>
</table>

<table>
<tr><td rowspan="3">Horizontal Asymptote</td><td colspan="2">look at Highest exponent on top & bottom</td></tr>
<tr><td>Bottom Heavy</td><td>Equally Weighted</td></tr>
<tr><td>$y=0$</td><td>ratio of leading coefficients</td></tr>
<tr><td>x-intercepts</td><td colspan="2">set numerator $=0$ (simplified)</td></tr>
<tr><td>y-intercept</td><td colspan="2">plug 0 in for x.</td></tr>
</table>